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CLOSED STREAMLINE FLOWS PAST SMALL ROTATING PARTICLES: HEAT TRANSFER AT HIGH PÉCLET NUMBERS

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Abstract—A heat transfer problem is solved, first for an infinitely long heated cylinder and then for a small heated sphere, each freely suspended in a general linear flow at Reynolds numbers $Re \ll 1$. Asymptotic solutions to the convection problem are developed for very large values of the Péclet number Pe, and expressions are obtained for the asymptotic Nusselt number for two-dimensional flows ranging from solid body rotation to hyperbolic flow. Since the objects in these cases are surrounded by a region of effectively isothermal closed streamlines, the asymptotic Nusselt number becomes independent of the Péclet number in the limit $Pe \to \infty$.

1. INTRODUCTION

Many empirical models have appeared in the literature, which relate the Nusselt number Nu to the Péclet number Pe and Reynolds number Re for a variety of flows, and which can be used to predict heat and mass transfer rates from spheres and cylinders suspended in low Reynolds number velocity fields. The theoretical effort has been directed to the cases of uniform flow and simple shear flow at infinity, and has led to the development of asymptotic expressions valid as $Pe \to 0$ and as $Pe \to \infty$. For low values of Pe, the asymptotic Nusselt number for the case of uniform flow is $O[(\ln Pe)^{-1}]$ for small cylinders and is equal to $2 + O(Pe)$ for small spheres. Similarly, for the case of a small particle freely rotating in a simple shear, the asymptotic Nusselt number for $Pe \to 0$ is still $O[(\ln Pe)^{-1})]$ for small cylinders but becomes $2 + O(Pe^{1/2})$ for small spheres. Thus, the main features of heat and mass transfer in low Reynolds number flow regimes can be inferred from these asymptotic expressions as $Pe \to 0$.

On the other hand, for asymptotically large Péclet numbers, Frankel & Acrivos (1968) showed that the Nusselt number for heat transfer from an isothermal cylinder freely rotating in a low Reynolds number simple shear becomes independent of the magnitude of the shear and approaches a constant value of 5.73. This result was then confirmed experimentally by Robertson & Acrivos (1970). Also, Acrivos (1971) solved the corresponding sphere problem using an approximate method and found that the asymptotic Nusselt number for $Pe \to \infty$ is 9. This is in contrast to the case of low Reynolds number uniform flow past a stationary particle where it is well-known that Nu becomes $O(Pe^{1/3})$ for asymptotically large Péclet numbers (cf. Acrivos & Taylor 1962). The distinguishing factor between these two cases is that for a simple shear, a freely rotating particle is completely surrounded by a region of isothermal closed streamlines, across which heat is transferred to the main stream by conduction alone, thereby insuring that the asymptotic Nusselt number is independent of Pe, as $Pe \to \infty$. But, in the case of uniform flow, where the streamlines emanate from upstream infinity, the transport of heat takes place, both by conduction and convection, across a thermal boundary layer of thickness $O(Pe^{-1/3})$ adjacent to the particle surface. Thus, at high Péclet numbers, heat transfer rates from a particle to a surrounding fluid depend primarily on the structure of the flow near the heated particle, i.e. they depend on whether the streamlines near the particle are open or closed.

So far, in the case of shear flows, attention has been directed to particles rotating in a simple shear. Clearly though, the corresponding heat transfer problem for a cylinder and a sphere freely rotating in any general linear flow would be worth investigating. This will be the aim of the

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present analysis which will be restricted to values of $Re \ll 1$ and will focus on the development of asymptotic solutions to the convection problem for $Pe \to \infty$.

It would be expected, of course, that the results to be presently derived would be similar to those found in the case of a simple shear, the difference being, though, that the Nusselt number should now depend directly on the parameters of the flow field. Specifically, denoting the freestream rate of strain tensor by e'_{ij} and the freestream vorticity vector by ω'_i, we would expect that, for the case of low Reynolds number flows, $Nu = f(e'_{ij}, \omega'_i)$ as $Pe \to \infty$, where Nu, being a scalar quantity, must be a function of the five scalar invariants of e'_{ij} and ω'_i:

$$I_1 = (e'_{ij}e'_{ij})^{1/2},\ I_2 = |\det e'_{ij}|^{1/3},\ I_3 = (e'_{ik}e'_{jk}\omega'_i\omega'_j)^{1/4},$$
$$I_4 = |e'_{ij}\omega'_i\omega'_j|^{1/3},\ I_5 = (\omega'_i\omega'_i)^{1/2}. \qquad [1.1]$$

Note that absolute values are used for I_2 and I_4, since the Nusselt number is invariant to flow reversal, and that each I_j has the same dimensions (units of reciprocal time). Moreover, since at low Reynolds numbers both the flow pattern and the asymptotic Nusselt number are not affected by multiplying the flow velocities (and hence, e'_{ij} and ω'_i) by a constant, Nu must be a function of the ratios of the invariants to, say I_5; or

$$Nu = f(J_1, J_2, J_3, J_4), \qquad [1.2]$$

where $J_i \equiv I_i/I_5$. As stated previously, at low Reynolds numbers, Nu is independent of the Péclet number only for those linear flows which produce a region of closed streamlines surrounding the particle, a condition that is satisfied if $I_5 \neq 0$.

2. STATEMENT OF THE PROBLEM

The dimensionless energy equation in Cartesian coordinates is

$$u_i \frac{\partial T}{\partial x_i} = \frac{1}{Pe} \frac{\partial^2 T}{\partial x_j \partial x_j}, \qquad [2.1]$$

where u_i is the velocity vector divided by $2\Omega a$, a is the radius of the cylinder or sphere in terms of which all lengths are rendered dimensionless, 2Ω is a characteristic freestream vorticity to be defined below, T is the dimensionless temperature and Pe is the Péclet number $Re\sigma$, with Re being the Reynolds number $2\Omega a^2/\nu$ and σ the Prandtl number. The boundary conditions are

$$T = 1 \quad \text{at} \quad r = 1,$$
$$T \to 0 \quad \text{at} \quad r \to \infty, \qquad [2.2]$$

where $r \equiv (x_i x_i)^{1/2}$. It can be seen from [2.1] that, as $Pe \to \infty$, the streamlines become isothermal, but since conduction is the primary mode of heat transport when the streamlines are closed and isothermal, it is necessary to take the conduction terms into account in solving [2.1] as $Pe \to \infty$. As shown by Acrivos (1971), this is accomplished by multiplying both sides of [2.1] by $\mathrm{d}t$, t denoting time, and integrating along a closed streamline. Thus,

$$\oint \frac{\partial^2 T}{\partial x_j \partial x_j} \mathrm{d}t = Pe \oint u_i \frac{\partial T}{\partial x_i} \mathrm{d}t = Pe \oint \frac{\mathrm{d}T}{\mathrm{d}t} \mathrm{d}t = 0, \qquad [2.3]$$

for all closed streamlines and for all Pe. Equation [2.3] will be used extensively in solving for the temperature distributions near the particle surface. However, the quantity of most interest here is the Nusselt number Nu which, based on the particle diameter $2a$, takes the forms

$$Nu = \frac{1}{\pi}\int_0^{2\pi} -\left(\frac{\partial T}{\partial r}\right)_{r=1} \mathrm{d}\varphi, \qquad [2.4]$$

for the cylinder, where cylindrical coordinates (r, φ) are used, and

$$Nu = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi} -\left(\frac{\partial T}{\partial r}\right)_{r=1} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi, \qquad [2.5]$$

for the sphere in terms of the spherical coordinates (r, θ, φ).

In a linear flow field, if the Reynolds number is sufficiently small to allow inertia effects to be neglected, the complete fluid velocity u_i must satisfy

$$\frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\partial p}{\partial x_i}, \qquad [2.6a]$$

$$\frac{\partial u_i}{\partial x_i} = 0, \qquad [2.6b]$$

with boundary conditions

$$u_i = \epsilon_{ijk}\Omega_j x_k \quad \text{at} \quad r = 1,$$

$$u_i \to e_{ij}x_j + \frac{1}{2}\epsilon_{ijk}\omega_j x_k \quad \text{as} \quad r \to \infty,$$

where p is the pressure divided by $2\Omega\mu$, e_{ij} the freestream rate of strain tensor divided by 2Ω, ω_j the freestream vorticity vector divided by 2Ω, Ω_j the angular velocity of the particle divided by 2Ω and $r \equiv (x_i x_i)^{1/2}$. Also, for a freely rotating cylinder or sphere, $\Omega_j = \frac{1}{2}\omega_j$. Thus, the boundary conditions become

$$u_i = \epsilon_{ijk}\Omega_j x_k \quad \text{at} \quad r = 1, \qquad [2.7a]$$

$$u_i \to e_{ij}x_j + \epsilon_{ijk}\Omega_j x_k \quad \text{as} \quad r \to \infty. \qquad [2.7b]$$

The scalar quantity Ω is defined as $\Omega \equiv (\Omega_i'\Omega_i')^{1/2} = \frac{1}{2}(\omega_i'\omega_i')^{1/2}$, where Ω_i' and ω_i' are, respectively, the dimensional angular velocity of the particle and the dimensional freestream vorticity vector. The quantity 2Ω can be thought of as a root square average vorticity. It is the quantity $2\Omega a$ by which all velocities have been rendered dimensionless. (The case $\Omega = 0$ is discussed later.) Note that $I_5 = 2\Omega$.

The present analysis will be restricted to free stream velocities which are two-dimensional, in which case J_2, J_3 and J_4 become zero (see [1.1–2]). Thus, the complete velocity distributions for both the cylinder and the sphere will contain only one flow parameter, namely $J_1 = I_1/I_5$. The functional dependence of the Nusselt number on this one flow parameter J_1 will now be obtained, first for an infinitely long cylinder and then for a sphere, both freely rotating in a two-dimensional linear flow field. Although still rather specialized, it is felt that the solution to these two-dimensional flow problems will be of some interest in that it will give further insight into the more complicated case of three-dimensional linear flows.

3. THE CYLINDER

In terms of the cylindrical coordinates (r, φ) and the two-dimensional streamfunction ψ, defined by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} \quad \text{and} \quad u_\varphi = -\frac{\partial \psi}{\partial r},$$

the Stokes solution, satisfying [2.7] together with the requirement that the net torque on the cylinder be zero, is

$$\psi = -\frac{1}{4}(r^2 - 1) - \frac{1}{4}A(r - r^{-1})^2 \cos 2\varphi, \qquad [3.1]$$

where $A \equiv \sqrt{2}J_1$ (note that $0 \le A \le \infty$). The solution for a simple shear as obtained by Cox *et al.* (1968) is easily recovered by setting $A = 1$ in [3.1]. The equation of a streamline may now be expressed in the form

$$k = r^2 + Ar^2(1 - r^{-2})^2 \cos 2\varphi, \qquad [3.2]$$

where $k \equiv 1 - 4\psi$, and the fluid velocity is given by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \varphi} = \frac{1}{2}Ar(1 - r^{-2})^2 \sin 2\varphi, \qquad [3.3a]$$

$$u_\varphi = -\frac{\partial \psi}{\partial r} = \frac{1}{2}r + \frac{1}{2}Ar(1 - r^{-4}) \cos 2\varphi. \qquad [3.3b]$$

There are two general classes of flows represented by [3.2], corresponding to two ranges of the flow parameter A; and moreover, the solution to the heat transfer problem is quite different in each case. Class I flows ($0 \le A < 1$) consist of flows having only closed streamlines, while Class II flows ($1 \le A \le \infty$) contain both closed and open streamlines.

All streamlines of Class I flows, including those in the free stream, are closed. This can easily be seen by letting $r \to \infty$ and rearranging [3.2] to give

$$r^2 \to \frac{k}{1 + A \cos 2\varphi} \quad \text{as} \quad r \to \infty. \qquad [3.4]$$

This is the equation of a family of ellipses for $0 \le A < 1$. Thus, the streamlines represented by [3.2] can be thought of as distorted ellipses, with those at infinity being undistorted. In fact, this is the situation regardless of the value of the Reynolds number, as long as the flow remains laminar, since the streamlines represented by boundary condition [3.4] are always closed provided that $0 \le A < 1$.

Class II flows, for which $1 \le A \le \infty$, have both closed and open streamlines (figure 1). The case $A = 1$ corresponds to a simple shear flow, where all streamlines are open except those

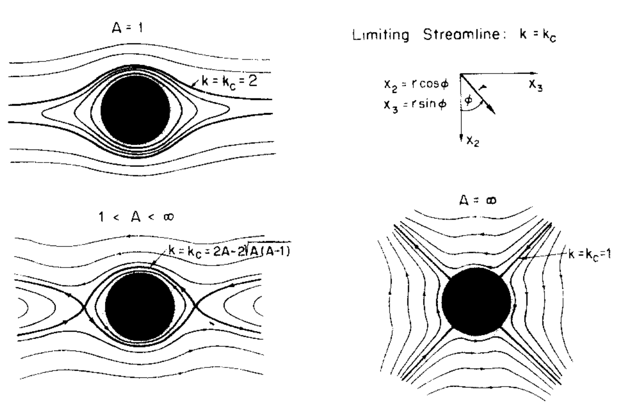

Figure 1. Streamlines around a cylinder freely rotating in a linear flow field when $A \ge 1$.

contained within the limiting streamline, which are all closed. For the $A = 1$ flow, which has been treated extensively by Cox *et al.* (1968), this closed streamline region extends to infinity. For the case $1 < A < \infty$, the closed streamline region is finite (figure 1), with the limiting streamline approaching the surface of the cylinder and the closed streamline region becoming smaller as $A \to \infty$. Thus, when $A = \infty$, the limiting streamline collapses onto the surface of the cylinder and no closed streamlines exist anywhere in the flow (this corresponds to pure straining motion or a zero vorticity field). The two stagnation points shown in figure 1 can be found by setting u_r and u_φ in [3.3] equal to zero. Denoting the location of these points by (r_s, φ_s), we find that

$$\varphi_s = \frac{\pi}{2}, \frac{3\pi}{2},$$

$$r_s = \left(\frac{A}{A-1}\right)^{1/4} \quad \text{for} \quad 1 \le A \le \infty.$$

Moreover, since the stagnation points are on the limiting streamline, we can obtain an expression for the limiting streamline $k = k_c$, by substituting r_s and φ_s into [3.2]. Thus,

$$k_c = 2A - 2\sqrt{[A(A-1)]} \quad \text{for} \quad 1 \le A \le \infty. \qquad [3.5]$$

Notice that $k_c = 2$ for $A = 1$ and $k_c = 1$ for $A = \infty$. Equation [3.2] can also be expressed as

$$r^2 = \frac{k + 2A\cos 2\varphi + \sqrt{[k^2 + 4A(k-1)\cos 2\varphi]}}{2(1 + A\cos 2\varphi)} \qquad [3.6]$$

where the plus sign is chosen in front of the radical since we are concerned only with the closed streamline region; i.e. [3.6] is valid only for $1 \le k \le k_c$ and $1 \le A \le \infty$.

Returning now to the heat transfer problem, we recall that [2.3] simplifies for two-dimensional flows and $Pe \gg 1$ (cf. Frankel & Acrivos 1968) to

$$\frac{d}{dk}\left[\Gamma(k)\frac{dT}{dk}\right] = 0, \qquad [3.7]$$

where k is related to the streamfunction ($k \equiv 1 - 4\psi$) and $\Gamma(k)$ is the circulation along a given streamline. Note that the temperature T is a constant along the streamline. The associated boundary conditions are

$$T = 1 \quad \text{at} \quad k = 1 \quad \text{(the surface of the cylinder)}, \qquad [3.8a]$$

$$T = 0 \quad \text{at} \quad k = k_c \quad \text{(the limiting streamline)}; \qquad [3.8b]$$

hence,

$$T = 1 - \frac{\int_1^k \Gamma^{-1}\, ds}{\int_1^{k_c} \Gamma^{-1}\, ds}. \qquad [3.9]$$

Then the Nusselt number, as determined from [2.4], becomes

$$Nu = \left[\frac{\pi}{4}\int_1^{k_c} \frac{dk}{\Gamma(k)}\right]^{-1}, \qquad [3.10]$$

where the circulation $\Gamma(k)$ is given by

$$\Gamma(k) = \int_0^{2\pi} \left(\frac{u_r^2 + u_\varphi^2}{u_\varphi}\right) r \, d\varphi, \qquad [3.11]$$

and where u_r, u_φ and r are determined as functions of k and φ from [3.3] and [3.6].

When $0 \le A < 1$ every streamline is closed, and so, $k_c = \infty$ for this case. Evaluating $\Gamma(k)$, as $k \to \infty$, from [3.11], we find that

$$\Gamma(k) \underset{k\to\infty}{\to} \frac{\pi k}{\sqrt{(1-A^2)}} + O(1) \quad \text{where} \quad 0 \le A < 1.$$

Hence, the integral in [3.10] has a logarithmic singularity and, therefore, $Nu = 0$ for $0 \le A < 1$. Of course, this result should have been expected in view of the fact that for two-dimensional closed streamline flows heat is confined to an effectively self-contained region surrounding the cylinder, i.e. *all* of the fluid is heated up to $T = 1$ (the cylinder surface temperature), and thus, never reaches a temperature of zero since no "free stream" region exists to which the heat emanating from the cylinder could be transferred by convection. Moreover, this seemingly paradoxical result might be expected to remain valid at any Reynolds number, since (for $0 \le A < 1$) all streamlines will be two-dimensional and closed far away from the cylinder owing to the nature of the undisturbed flow.

For $1 \le A \le \infty$, the integral in [3.10] was evaluated numerically for various values of A, and the results are plotted in figure 2. As reported earlier by Frankel & Acrivos (1968), Nu was found to equal 5.73 for $A = 1$.

As depicted in figure 2, the Nusselt number becomes linear in A as $A \to \infty$. The appropriate asymptotic formula can be obtained by expanding [3.6], the expression for r^2, in reciprocal powers of A as $A \to \infty$, thereby determining $\Gamma(k)$ in powers of $(1/A)$, which is then substituted into [3.10] and integrated. It is found that

$$Nu \to 15.58A - 5.93 + O(A^{-1}) \quad \text{as} \quad A \to \infty. \qquad [3.12]$$

Agreement between the exact value of Nu as computed numerically from [3.10] and the asymptotic Nusselt number from [3.12] is very good even for A as low as 1.2, where the asymptotic Nu exceeds the true result by only 16%. Thus, the asymptotic expression [3.12] represents a very satisfactory approximation to the true Nusselt number over a wide range of A values.

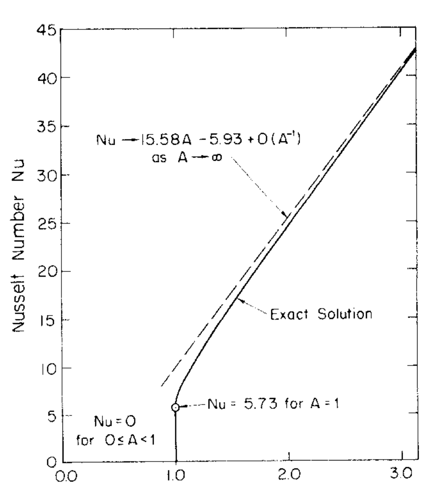

Figure 2. The Nusselt number for a heated cylinder freely rotating in a linear flow field.

If $A \equiv \sqrt{2}J_1 = \infty$ then $I_5 \equiv 2\Omega = 0$ (see [1.1–2]), and the undisturbed flow corresponds to a pure straining motion. Thus, there are no closed streamlines and the Nusselt number is infinite. In fact, as $Pe \to \infty$, there exists a thermal boundary layer adjacent to the cylinder surface having a thickness $O(Pe^{-1/3})$. Using this fact, it can be shown (Poe, 1975) that when $A = \infty$,

$$Nu \to 1.46\, Pe^{1/3} \quad \text{as} \quad Pe \to \infty. \qquad [3.13]$$

Thus, only when $J_1 = \infty$ is the Nusselt number dependent on the Péclet number for the type of problems being considered here.

4. THE SPHERE

For a sphere in a two-dimensional linear flow in the x_2–x_3 plane, the Stokes solution satisfying [2.7] plus the condition of zero torque is given in Cartesian coordinates by

$$u_i = \frac{1}{2}\,\delta_{i2}x_3(A-1) + \frac{1}{2}\,\delta_{i3}x_2(A+1) + \frac{1}{2}A\left[\frac{5x_ix_2x_3(1-r^2)}{r^7} - \frac{(\delta_{i2}x_3 + \delta_{i3}x_2)}{r^5}\right], \qquad [4.1]$$

where again $A \equiv \sqrt{2}J_1$ $(0 \le A \le \infty)$. In terms of the spherical coordinates (r, θ, φ) shown in figure 3, the velocity distribution then becomes

$$u_r = \frac{1}{2}Ar\left(1 - \frac{5}{2}r^{-3} + \frac{3}{2}r^{-5}\right)\sin^2\theta \sin 2\varphi, \qquad [4.2a]$$

$$u_\theta = \frac{1}{2}Ar(1 - r^{-5})\sin\theta\cos\theta\sin 2\varphi, \qquad [4.2b]$$

$$u_\varphi = \frac{1}{2}r\sin\theta + \frac{1}{2}Ar(1 - r^{-5})\sin\theta\cos 2\varphi, \qquad [4.2c]$$

which, for $A = 1$, reduces to the solution for a simple shear as obtained by Cox *et al.* (1968). It can easily be shown (Poe 1975) that the streamlines are formed by the intersection of the two sets of surfaces,

$$x_1 = r\cos\theta = Crf(r), \qquad [4.3a]$$

$$x_2 = r\sin\theta\cos\varphi = \pm rf(r)\left[E + g(r) + \left(1 - \frac{1}{A}\right)h(r)\right]^{1/2} \quad (A \ne 0), \qquad [4.3b]$$

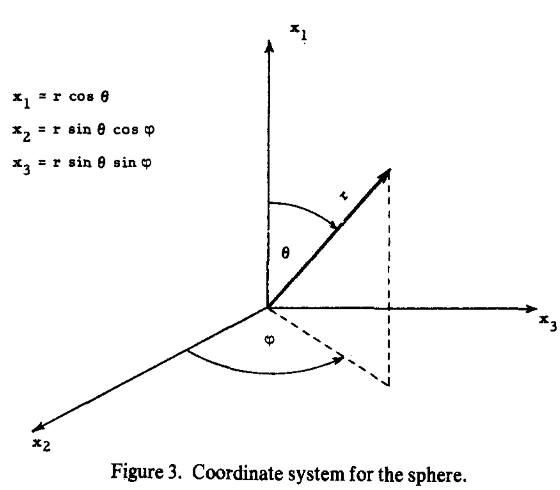

Figure 3. Coordinate system for the sphere.

where C and E are parameters and

$$f(r) \equiv \left(r^3 - \frac{5}{2} + \frac{3}{2} r^{-2}\right)^{-1/3}, \qquad [4.4a]$$

$$g(r) \equiv \int_r^\infty y^{-3} f(y)\, dy, \qquad [4.4b]$$

$$h(r) \equiv \int_1^r y^2 f(y)\, dy. \qquad [4.4c]$$

Furthermore, f, g and h possess the following properties: as $r \to 1$,

$$f(r) \sim \left(\frac{2}{15}\right)^{1/3} (r-1)^{-2/3} \left[1 + \frac{2}{9}(r-1) - \frac{19}{81}(r-1)^2 + O(r-1)^3\right], \qquad [4.5a]$$

$$g(r) \sim g(1) - 3\left(\frac{2}{15}\right)^{1/3} (r-1)^{1/3} \left[1 - \frac{25}{36}(r-1) + \frac{59}{81}(r-1)^2 + O(r-1)^3\right], \qquad [4.5b]$$

$$h(r) \sim 3\left(\frac{2}{15}\right)^{1/3} (r-1)^{1/3} \left[1 + \frac{5}{9}(r-1) + \frac{14}{81}(r-1)^2 + O(r-1)^3\right]; \qquad [4.5c]$$

while, as $r \to \infty$,

$$f(r) \sim r^{-1} + \frac{5}{6} r^{-4} + O(r^{-6}), \qquad [4.6a]$$

$$g(r) \sim \frac{1}{3} r^{-3} + \frac{5}{36} r^{-6} + O(r^{-8}), \qquad [4.6b]$$

$$h(r) \sim \frac{1}{2} r^2 + O(1) + O(r^{-1}). \qquad [4.6c]$$

Also, C must lie between $-C^*$ and $+C^*$, where

$$C^* f(r^*) = 1 \quad \text{with} \quad E + g(r^*) + \left(1 - \frac{1}{A}\right) h(r^*) = 0.$$

The streamlines represented by [4.3] are in general three-dimensional but for the case $C = 0$ they become coplanar, all lying in the x_2–x_3 plane.

As in the case of the cylinder, there are two general classes of flows represented by [4.3], corresponding to two ranges of the flow parameter A; and again, the solution to the heat transfer problem is quite different in each case. Class I flows ($0 \le A < 1$) consist of flows having only closed streamlines, while Class II flows ($1 \le A \le \infty$) contain both closed and open streamlines.

Let us first then consider Class I flows for which all streamlines are closed. Here, the undisturbed flow consists of a family of elliptic cylinders with the x_1 axis as their central axis, i.e. in the plane x_1 = constant, the undisturbed streamlines are a family of ellipses. Letting $r \to \infty$ in [4.3] and rearranging, we find that

$$x_2^2 + x_3^2 \to \frac{B}{1 + A \cos 2\varphi} \quad \text{as} \quad r \to \infty \quad (A \neq 0), \qquad [4.7]$$

which is just the equation of a family of ellipses in the x_1 = constant plane. (The constant $B \equiv 2AE - C^2(1-A)$ is positive for any given streamline.) Thus, when $0 < A < 1$, the streamlines at infinity are undistorted, planar ellipses. In fact, this is the case regardless of the value of the

Reynolds number, since the streamlines represented by boundary condition [4.7] are always closed provided that $0 < A < 1$.

Class II flows, for which $1 \leq A \leq \infty$, contain both closed and open streamlines. The case $A = 1$, which was considered in detail by Cox *et al.* (1968), corresponds to a simple shear flow at infinity and consists entirely of open streamlines except those contained within a three-dimensional limiting streamsurface, which are all closed. This closed streamline region extends to infinity for $A = 1$. As in the case of the cylinder, this closed streamline region is finite for $1 < A < \infty$, with the limiting streamsurface approaching the surface of the sphere and the closed streamline region becoming smaller as $A \to \infty$. And when $A = \infty$, the limiting streamsurface collapses onto the surface of the sphere and there are no closed streamlines anywhere in the flow (this corresponds to an undisturbed flow having zero vorticity).

When $1 \leq A \leq \infty$, there is a locus of points (a circle, in fact) in the $x_1 - x_3$ plane where u_r and u_φ vanish. Denoting the location of these points by (r_s, φ_s) and setting u_r and u_φ equal to zero in [4.2a,c] we find that

$$\varphi_s = \frac{\pi}{2}, \frac{3\pi}{2}, \qquad [4.8a]$$

$$r_s = \left(\frac{A}{A-1}\right)^{1/5}. \qquad [4.8b]$$

Also, it is obvious that these points lie on the limiting streamsurface. Denoting the latter by $E = E_c$, we can deduce from [4.3b] that

$$E_c = -g(r_s) - \left(1 - \frac{1}{A}\right) h(r_s), \qquad [4.9]$$

where r_s is given by [4.8b]. Notice that when $A = 1$, $E_c = 0$ and when $A = \infty$, $E_c = -g(1)$. Also, one can easily see that $E = -g(1)$ corresponds to the surface of the sphere and that for $1 \leq A \leq \infty$ all closed streamlines are contained within the space lying between the sphere and the limiting three-dimensional stream surface, $E = E_c$. For $E > E_c$ all streamlines are open.

Returning now to the heat transfer problem, we note that for the case $A = 0$ the sphere is undergoing pure rotation (u_r and u_θ are zero); and so, the temperature distribution is identical to that for pure conduction, or $T = 1/r$. Thus, for $A = 0$, $Nu = 2$. An attempt was made to solve the heat equation [2.1] for the case $A \ll 1$ by means of a regular perturbation expansion about $A = 0$, but no definitive results could be obtained, perhaps owing to the fact that the leading term of the expansion, $T = 1/r$, may not represent a uniformly valid approximation to the temperature field as $A \to 0$. A different approach, therefore, would have been required, but in view of the expected difficulties in constructing such a solution and the fact that results already exist for $A = 0$ and $A = 1$, it was felt advisable to proceed directly to the class of flows for which $1 \leq A \leq \infty$.

Let us next consider the heat transfer problem for the case $1 \leq A \leq \infty$. We note that since the temperature along any given closed streamline is a constant in the limit $Pe \to \infty$, and since E and C are constant along a streamline, it is evident that the temperature T is a function only of E and C. Furthermore, the heat transfer rate at high Pe is determined primarily by the structure of the flow near the surface of the sphere. So, following an analysis similar to that for a simple shear as presented by Acrivos (1971), we shall make use of these results, together with [2.3], to obtain an equation for the temperature distribution near the surface of the sphere as $Pe \to \infty$.

It can be shown first of all (Poe 1975) that the temperature T satisfies the differential equation

$$a_1 \frac{\partial^2 T}{\partial \xi^2} + a_2 \frac{\partial^2 T}{\partial \xi \partial \eta} + a_3 \frac{\partial^2 T}{\partial \eta^2} + a_4 \frac{\partial T}{\partial \xi} + a_5 \frac{\partial T}{\partial \eta} = 0, \qquad [4.10]$$

where

$$a_1 \equiv \xi^{-4}\left\{1 - \left[5A\eta^2 - \frac{20}{9}\right]\xi^3 + \left[A^2\left(\frac{275}{16}\eta^4 - \frac{25}{6}\eta^2 - \frac{25}{144}\right) - \frac{55}{9}A\eta^2 + \frac{26}{27}\right]\xi^6 + O(\xi^9)\right\},$$

$$a_2 \equiv 4\eta\xi^{-5}\left\{\left[\frac{5}{4}A\eta^2 - \frac{2}{3}\right]\xi^3 - \left[A^2\left(\frac{225}{32}\eta^4 - \frac{125}{48}\eta^2 + \frac{25}{96}\right) - \frac{10}{3}A\eta^2 + \frac{2}{3}\right]\xi^6 + O(\xi^9)\right\},$$

$$a_3 \equiv 4\eta^2\xi^{-6}\left\{\left[A^2\left(\frac{75}{32}\eta^4 - \frac{25}{16}\eta^2 + \frac{25}{32}\right) - \frac{5}{3}A\eta^2 + \frac{7}{36}\right]\xi^6 + O(\xi^9)\right\},$$

$$a_4 \equiv -2\xi^{-5}\left\{1 - \left[\frac{5}{4}A\eta^2 + \frac{4}{9}\right]\xi^3 + \left[A^2\left(\frac{25}{32}\eta^2 + \frac{25}{48}\eta^2 + \frac{175}{288}\right) + \frac{20}{9}A\eta^2 - \frac{34}{27}\right]\xi^6 + O(\xi^9)\right\},$$

$$a_5 \equiv -2\eta\xi^{-6}\left\{-\left[A^2\left(\frac{225}{32}\eta^4 - \frac{75}{16}\eta^2 + \frac{75}{32}\right) - \frac{10}{3}A\eta^2 - \frac{11}{9}\right]\xi^6 + O(\xi^9)\right\},$$

with $\xi \equiv (5/6)^{1/3}A\lambda$, $\eta \equiv (6/5)C(A\lambda)^{-2}$ and $\lambda \equiv E + g(1) = E + 1.047$. Strictly speaking, the above expressions for the coefficients a_i are valid only close to the surface of the sphere; however, in order to solve [4.10] we shall resort to an approximate method whereby we shall retain these expressions throughout the closed streamline region. Although it is obvious that the approximation should become better as additional terms in the coefficients of [4.10] are retained, a fairly good estimate of T can still be computed using just the terms that are shown.

This equation must now be solved with the following boundary conditions,

$$T = 1 \quad \text{at} \quad \xi = 0 \quad \text{(surface of sphere)}, \qquad [4.11a]$$

$$T = 0 \quad \text{at} \quad \xi = \xi_c \quad \text{(limiting streamsurface)}, \qquad [4.11b]$$

$$T \text{ is finite for } 0 \le \xi \le \xi_c \text{ and } -\eta^* \le \eta \le +\eta^*, \qquad [4.12a]$$

where

$$\eta^* = \left\{1 + \left(\frac{5}{6}A - \frac{4}{9}\right)\xi^3 + \left(\frac{25}{16}A^2 - \frac{5}{3}A + \frac{10}{27}\right)\xi^6 + O(\xi^9)\right\}, \qquad [4.12b]$$

and where $\xi_c = (5/6)^{1/3}A[E_c + g(1)]$. Using the expression for E_c given in [4.8b] and [4.9], ξ_c^3 was computed numerically for various values of A $(1 \le A \le \infty)$ and the results are presented in figure 4. It can be seen from figure 4 that as $A \to \infty$, the limiting streamsurface approaches the surface of the sphere ($\xi = 0$) as expected. Also, the asymptotic forms of ξ_c^3 as $A \to 1$ and as $A \to \infty$ can be

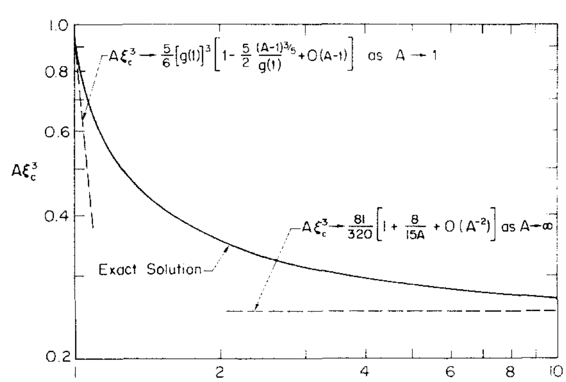

Figure 4. The critical streamsurface for a freely rotating sphere in a linear flow field when $A \ge 1$.

computed using [4.5] and [4.6]. These are given by

$$\text{as} \quad A \to 1, \quad \xi_c^3 \to \frac{5}{6}[g(1)]^3\left\{1 - \frac{5}{2}\frac{(A-1)^{3/5}}{g(1)} + O(A-1)\right\}, \tag{4.13}$$

$$\text{as} \quad A \to \infty, \quad \xi_c^3 \to \frac{81}{320}A^{-1}\left\{1 + \frac{8}{15A} + O(A^{-2})\right\}. \tag{4.14}$$

The differential equation [4.10] may be solved in a variety of ways. The technique employed here will, up to a point, parallel that used by Acrivos (1971) in solving the related equation for a simple shear ($A = 1$). We can proceed by first assuming that the temperature distribution can be expressed as,

$$T = 1 - \alpha_1(\eta)\xi^3 + \alpha_2(\eta)\xi^6 - \alpha_3(\eta)\xi^9 + \cdots, \tag{4.15}$$

which, when substituted in [4.10], leads to the recursion relations

$$\alpha_2 = \left(\frac{5}{6}A\eta^2 - \frac{4}{9}\right)\eta\frac{d\alpha_1}{d\eta} - \left(\frac{5}{4}A\eta^2 - \frac{8}{9}\right)\alpha_1, \tag{4.16}$$

$$\begin{aligned}\alpha_3 = \frac{1}{9}\Bigg\{&\left[\frac{25}{48}A^2(5\eta^4 + 2\eta^2 - 1) - \frac{10}{3}A\eta^2 + \frac{19}{18}\right]\eta^2\frac{d^2\alpha_1}{d\eta^2} \\ &- \left[\frac{25}{96}A^2(3\eta^4 + 14\eta^2 + 1) - \frac{40}{3}A\eta^2 + \frac{43}{9}\right]\eta\frac{d\alpha_1}{d\eta} \\ &- \left[\frac{25}{32}A^2(\eta^4 - 6\eta^2 - 1) + 20A\eta^2 - \frac{76}{9}\right]\alpha_1\Bigg\}, \text{ etc.}\end{aligned} \tag{4.17}$$

Next, we consider the result of truncating the series [4.15] at successively higher terms and applying the boundary condition $T = 0$ at $\xi = \xi_c$. Retaining only two terms of [4.15], or

$$T = 1 - \alpha_1(\eta)\xi^3,$$

we find that $\alpha_1^{(1)} = \xi_c^{-3}$, where $\alpha_1^{(1)}$ denotes the first approximation to α_1. If three terms are retained, we find that

$$\alpha_1^{(2)} = \xi_c^{-3} + \alpha_2^{(1)}\xi_c^3.$$

Now we obtain $\alpha_2^{(1)}$ from [4.16] using the previously computed value of α_1, i.e. $\alpha_1^{(1)}$. This calculation gives a second approximation to α_1, or

$$\alpha_1^{(2)} = \xi_c^{-3} - \left(\frac{5}{4}A\eta^2 - \frac{8}{9}\right).$$

And finally, when four terms are retained in [4.15], we have

$$\alpha_1^{(3)} = \xi_c^{-3} + \alpha_2^{(2)}\xi_c^3 - \alpha_3^{(1)}\xi_c^6,$$

where $\alpha_2^{(2)}$ is computed from [4.16] using $\alpha_1^{(2)}$ for α_1 and $\alpha_3^{(1)}$ from [4.17] using $\alpha_1^{(1)}$ for α_1. So a third approximation to α_1 is given by

$$\alpha_1^{(3)} = \xi_c^{-3} - \left(\frac{5}{4}A\eta^2 - \frac{8}{9}\right) - \left[\frac{25}{288}A^2(5\eta^4 + 6\eta^2 + 1) - \frac{10}{9}A\eta^2 + \frac{4}{27}\right]\xi_c^3.$$

From [2.5], it is easy to show that the Nusselt number is

$$Nu = 3\int_{-1}^{+1} \alpha_1(\eta)\,d\eta;$$

thus, the first three approximations to Nu become

$$Nu^{(1)} = 6\xi_c^{-3}, \quad [4.18]$$

$$Nu^{(2)} = 6\xi_c^{-3} - \left(\frac{5}{2}A - \frac{16}{3}\right), \quad [4.19]$$

$$Nu^{(3)} = 6\xi_c^{-3} - \left(\frac{5}{2}A - \frac{16}{3}\right) - \left(\frac{25}{12}A^2 - \frac{20}{9}A + \frac{8}{9}\right)\xi_c^3. \quad [4.20]$$

Therefore, to $O(\xi_c^6)$, the Nusselt number is given in the limit of large Pe and for $1 \le A \le \infty$ by

$$Nu = 6\xi_c^{-3}\left\{1 - \left(\frac{5}{12}A - \frac{8}{9}\right)\xi_c^3 - \left(\frac{25}{72}A^2 - \frac{10}{27}A + \frac{4}{27}\right)\xi_c^6 + O(\xi_c^9)\right\}, \quad [4.21]$$

if the Reynolds number is sufficiently small for inertia effects to be negligible.

Table 1 gives the first three approximations to the Nusselt number using [4.18], [4.19] and

Table 1. Approximate and estimated values of Nu for a sphere when $A \ge 1$

A	$Nu^{(1)}$ computed from [4.18]	$Nu^{(2)}$ computed from [4.19]	$Nu^{(3)}$ computed from [4.20]	Best estimate of true value of Nu (see text)
1.00	6.27	9.11	8.39	8.9
1.04	8.41	11.1	10.6	11.0
1.07	9.59	12.2	11.7	12.1
1.13	11.6	14.1	13.5	13.9
1.20	13.6	16.0	15.4	15.8
1.40	19.0	20.8	20.2	20.6
1.70	26.5	27.6	26.9	27.3
2.00	33.9	34.2	33.4	33.9
3.00	58.0	55.8	54.4	54.7
10.00	224.0	205.0	200.0	200.0

[4.20] with selected values of A ($1 \le A \le \infty$). Shown in the last column of Table 1 are the best estimates of the true values of Nu based on the first three approximations. For each A in table 1, this estimate was found by plotting the first three approximations vs $1/M$, where M is the number of the iteration, and graphically extrapolating. It is interesting to note that the values computed from $Nu^{(2)}$ [4.19] agree quite well with the last column of numbers given in table 1, except possibly for very large values of A. These estimates of Nu are plotted in figure 5 for $A \ge 1$. It is seen here that Nu approaches asymptotically a straight line as $A \to \infty$. A good estimate of the asymptotic form of this line can be found by substituting [4.14] into [4.21]. This gives

$$Nu \to 20.7A - 7.0 + O(A)^{-1} \quad \text{as} \quad A \to \infty. \quad [4.22]$$

Again, as in the case of the cylinder, the asymptotic expression for Nu as $A \to \infty$ remains accurate even for as low a value of A as 1.2, where [4.22] overestimates the true value of Nu by only 13%.

If $A \equiv \sqrt{2}\,J_1 = \infty$, then $I_5 \equiv 2\Omega = 0$, and the undisturbed flow becomes irrotational. In fact, there are no closed streamlines and thus, the Nusselt number is infinite. For this case, as $Pe \to \infty$,

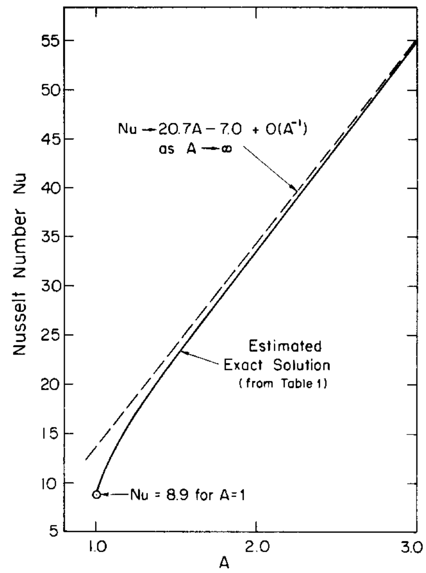

Figure 5. The Nusselt number for a heated sphere freely rotating in a linear flow field when $A \geq 1$.

there exists a thin thermal boundary layer $O(Pe^{-1/3})$ in thickness next to the surface of the sphere. It can be shown (Poe 1975) that when $A = \infty$,

$$Nu \rightarrow 1.60 Pe^{1/3} \quad \text{as} \quad Pe \rightarrow \infty. \qquad [4.23]$$

Thus, as in the case of the cylinder, the Nusselt number is a function of the Péclet number only when $J_1 = \infty$ (or $I_5 = 0$), if the Reynolds number is sufficiently small for inertia effects to be neglected.

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